Particle in a 1-D Box: TZDII[1] Ch. 7 Example 2

Consider a particle in the ground state (n = 1) of a rigid box of length a with a wave function

$$\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad \text{TZDII (7.60)}$$

a) Find the probability density $|\psi|^2$

b) Where is the particle most likely to be found?

c) What is the probability of finding the particle in the interval $0.50a \le x \le 0.51a$?

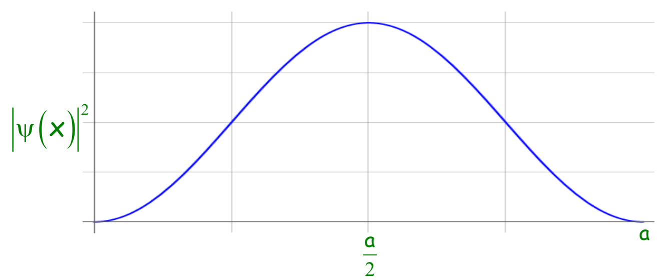
d) What is the probability of finding the particle in the interval $0.75a \le x \le 0.76a$?

e) What would be the average result if the position of a particle in the ground state were measured many times?

Repeat for a particle in the first excited state (n = 2).

a) The probability density $|\psi|^2$, for n = 1 is

$$|\psi(x)|^2 = \frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right)$$



b) Plotting this function gives the probability distribution that shows the most likely place to find the particle is at a/2. But it can also be found analytically as the maximum of the function:

$$\left.\frac{d|\psi(x)|^2}{dx}\right|_{x_{mp}} = 0$$

$$\left.\frac{d|\psi(x)|^2}{dx}\right|_{x_{mp}} = \frac{d}{dx}\left[\frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right)\right] = \frac{2}{a}\left[2\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\left(\frac{\pi}{a}\right)\right]\Bigg|_{x_{mp}} = 0$$

$$\frac{4\pi}{a^2}\left[\sin\left(\frac{\pi x_{mp}}{a}\right)\cos\left(\frac{\pi x_{mp}}{a}\right)\right] = 0$$

$$\Rightarrow\ x_{mp} = \frac{a}{2} \text{ for } \sin\left(\frac{\pi x_{mp}}{a}\right) = 0 \text{ which is the maximum}$$

$$\left(\text{or } x_{mp} = 0 \text{ for } \cos\left(\frac{\pi x_{mp}}{a}\right) = 0 \text{ which is the minimum}\right)$$

This confirms that $x_{mp} = a/2$.

[1] Modern Physics for Scientists and Engineers, 2nd Ed., John R. Taylor, Chris D. Zafiratos, & Michael A. Dubson (Prentice Hall, 2002)

c & d) What is the probability of finding the particle in the interval $0.50a \le x \le 0.51a$ and $0.75a \le x \le 0.76a$?

In general, the probability of finding the particle between x & x + dx is

$$\left|\psi(x)\right|^2 dx = P(\text{finding the particle between } x \text{ and } x+dx).$$

So the probability of finding it in a region between x_1 and x_2 is the integral which is approximated by a simple product for a small region:

$$\text{Prob. between } x_1 \ \& \ x_2 = \int_{x_1}^{x_2} \left|\psi(x)\right|^2 dx \approx \left|\psi(x = x_1)\right|^2 \Delta x$$

Thus for $x = 0.5a$ and $\Delta x = 0.01a$,

$$\text{Prob. between } 0.5a \ \& \ 0.51a \approx \left|\psi(0.5a)\right|^2 0.01a$$

$$\text{Prob. between } 0.5a \ \& \ 0.51a \approx \left(\frac{2}{a}\sin^2\left(\frac{\pi(0.5a)}{a}\right)\right)0.01a = 0.02\sin^2\left(\frac{\pi}{2}\right) = 0.02 = 2\%$$

$$\text{Prob. between } 0.75a \ \& \ 0.76a \approx \left(\frac{2}{a}\sin^2\left(\frac{\pi(0.75a)}{a}\right)\right)0.01a = 0.02\sin^2\left(\frac{3\pi}{4}\right) = 0.01 = 1\%$$

e) The average result if the position of a particle in the ground state were measured many times is the expectation value, the integral of the product of the function we want the expectation value of and the probability increment of it. Here, the function is the position and the probability density gives its probability increment

$$\langle f(x)\rangle = \int f(x)\, p(x)dx \Rightarrow \ \langle x\rangle = \int_0^a x\left(\frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right)\right)dx$$

CRC integral #282 is
$$\int x\sin^2(ax)dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

Giving
$$\langle x\rangle = \int_0^a x\left(\frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right)\right)dx = \frac{2}{a}\left[\frac{x^2}{4} - \frac{x\sin 2\left(\frac{\pi x}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos 2\left(\frac{\pi x}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right]_0^a$$

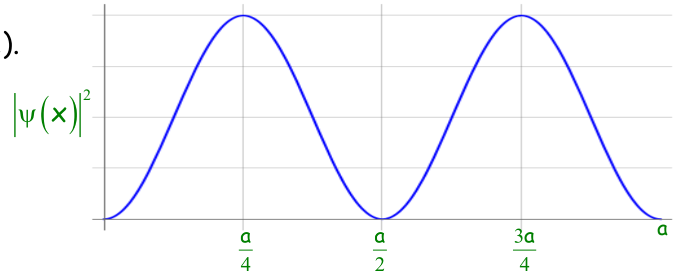
$$\langle x\rangle = \frac{2}{a}\left[\frac{a^2}{4} - \frac{a\sin 2\left(\frac{\pi a}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos 2\left(\frac{\pi a}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right] - \left[\frac{0^2}{4} - \frac{0\sin 2\left(\frac{\pi 0}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos 2\left(\frac{\pi 0}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right]$$

$$\langle x\rangle = \frac{2}{a}\left[\frac{a^2}{4} - \frac{a\sin 2\pi}{4\left(\frac{\pi}{a}\right)} - \frac{\cos 2\pi}{8\left(\frac{\pi}{a}\right)^2}\right] - \left[\frac{0^2}{4} - \frac{0\sin 2\left(\frac{\pi 0}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos 2\left(\frac{\pi 0}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right]$$

$$\langle x\rangle = \frac{2}{a}\left[\frac{a^2}{4} - \frac{a^2}{8\pi^2}\right] - \left[-\frac{a^2}{8\pi^2}\right] = \frac{a}{2}$$

Where the value of the integral turns out to be $\frac{a}{2}$ (problem 7.33). This means that the most likely place to find the particle is at 0.5a, which is what we said in part b! This is the analytical way to get this.

Repeat for a particle in the first excited state (n = 2).



a) The probability density $|\psi|^2$, for n = 2 is

$$|\psi(x)|^2 = \frac{2}{a}\sin^2\left(\frac{2\pi x}{a}\right)$$

b) This plot shows the particle is most likely to be found at either 0.25a or 0.75a.

c & d) What is the probability of finding the particle in the interval $0.50a \le x \le 0.51a$ and $0.75a \le x \le 0.76a$?

$$\text{Prob. between } 0.5a \text{ \& } 0.51a \approx \left(\frac{2}{a}\sin^2\left(\frac{2\pi(0.5a)}{a}\right)\right)0.01a = 0.02\sin^2(\pi) = 0$$

$$\text{Prob. between } 0.75a \text{ \& } 0.76a \approx \left(\frac{2}{a}\sin^2\left(\frac{2\pi(0.75a)}{a}\right)\right)0.01a = 0.02\sin^2\left(\frac{3\pi}{2}\right) = 0.02 = 2\%$$

so the actual probability of finding the excited particle at 0.25a or 0.75a is 2%, just like the probability of finding it at 0.5a for the ground state.

e) The expectation value is

$$\langle x \rangle = \int_0^a x\left(\frac{2}{a}\sin^2\left(\frac{2\pi x}{a}\right)\right)dx = \frac{a}{2}$$

Curious that the expectation value is where there is zero probability of finding the particle!! This shows the problem with averages … since it's equally likely to be at 0.25a and 0.75a, the average or expectation value puts it at 0.5a where it can never actually be found! With one foot in boiling water and one in ice water, on average, a person is quite comfortable. Yeah.